Flight Dynamics & Control
Modal Analysis

Harry G. Kwatny
Department of Mechanical Engineering & Mechanics
Drexel University

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Outline

- Similarity Transformations
- Eigenvalues & Eigenvectors
- Modal Coordinates
- Real & Complex Modes
- Lateral Dynamics: Dutch roll, roll and spiral modes
- Longitudinal Dynamics: Phugoid and Short-period modes

Similarity Transformations

$$\dot{x} = Ax + Bu \qquad \dot{x} = Ax + bu$$
$$y = Cx + Du \qquad x \in R^n, u \in R^m, y \in R^p \qquad y = cx + du$$

Now consider the transformation to new states *z*, defined by

$$x = Tz \Leftrightarrow z = T^{-1}x$$

$$\begin{matrix} T\dot{z} = ATz + Bu \\ y = CTz + Du \end{matrix} \Rightarrow \begin{matrix} \dot{z} = T^{-1}ATz + T^{-1}Bu \\ y = CTz + Du \end{matrix}$$

so that,

$$\begin{matrix} \dot{z} = A^* z + B^* u \\ y = C^* z + D^* u \end{matrix}, \quad A^* = T^{-1}AT, B^* = T^{-1}B, C^* = CT, D^* = D$$

Eigenvalues & Eigenvectors

Consider the square $n \times n$ matrix A as a map from R^n to R^n, i.e.,

$$y = Ax$$

Does there exist a nontrivial input vector h, such that the output vector y, points in the same direction as h, i.e., $y = \lambda h$, λ where is some scalar?

$$Ah = \lambda h$$

Let's try to solve for h

$$[\lambda I - A]h = 0 \Rightarrow \text{A nontrivial solution exists iff } \det[\lambda I - A] = 0.$$

Characteristic polynomial $\phi(\lambda) \triangleq \det[\lambda I - A] = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$

$\phi(\lambda) = 0$ has n roots $\{\lambda_1, \ldots, \lambda_n\}$, possibly complex, possibly repeated.

Choose one, say λ_i, $i \in \{\lambda_1, \ldots, \lambda_n\}$ and find the corresponding h_i

$$[\lambda_i I - A]h_i = 0$$

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Example 1 (distinct roots)

$$A=\begin{bmatrix}-5 & 3\\ 3 & -5\end{bmatrix}\Rightarrow \det\left[\lambda I-A\right]=\det\begin{bmatrix}\lambda+5 & -3\\ -3 & \lambda+5\end{bmatrix}=\left(\lambda+5\right)^2-9$$

$$\phi(\lambda)=\lambda^2+10\lambda+16=(\lambda+2)(\lambda+8)\Rightarrow \boxed{\lambda_1=-2,\lambda_2=-8}$$

$$\left[-2I-A\right]h_1=0\Rightarrow\begin{bmatrix}-2+5 & -3\\ -3 & -2+5\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=0$$

$$\begin{aligned}3\eta_1-3\eta_2&=0\\ -3\eta_1+3\eta_2&=0\end{aligned}\Rightarrow \eta_1=\eta_2\Rightarrow \boxed{h_1=\begin{bmatrix}1\\ 1\end{bmatrix}\eta_2}$$

$$\left[-8I-A\right]h_1\Rightarrow \begin{aligned}-3\eta_1-3\eta_2&=0\\ -3\eta_1-3\eta_2&=0\end{aligned}\Rightarrow \eta_1=-\eta_2\Rightarrow \boxed{h_2=\begin{bmatrix}-1\\ 1\end{bmatrix}\eta_2}$$

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Example 2 (repeated roots)

$$A = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} \Rightarrow \det[\lambda I - A] = \det \begin{bmatrix} \lambda + 5 & 0 \\ 0 & \lambda + 5 \end{bmatrix} = (\lambda + 5)^2$$

$$\phi(\lambda) = (\lambda + 5)(\lambda + 5) \Rightarrow \boxed{\lambda_1 = -5, \lambda_2 = -5}$$

$$[-5I - A]h_1 = 0 \Rightarrow \begin{bmatrix} -5+5 & 0 \\ 0 & -5+5 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \text{ satisfied for any } \eta_1, \eta_2$$

$$h_{1,2} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \quad \text{Choose} \quad \boxed{h_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, h_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Diagonal Form

eigen-system of A: $\begin{matrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \leftarrow \text{eigenvalues} \\ h_1 & h_2 & \cdots & h_n \leftarrow \text{independent eigenvectors} \end{matrix}$

$$T \triangleq \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}$$

$$\Rightarrow A^* = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{-1} A \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{-1} \begin{bmatrix} Ah_1 & Ah_2 & \cdots & Ah_n \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 h_1 & \lambda_2 h_2 & \cdots & \lambda_3 h_n \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{-1} \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$= diag\left(\lambda_1, \ldots \lambda_n\right)$$

$$\boxed{\dot{z}_i = \lambda_i z_i + b_i^* u,\ i = 1, \ldots, n}$$

A decoupled system of n 1st order ode's

Example

Define A

Compute eigensystem

Check similarity trans

Use linear solve rather than inv

```
>> A=[3 2 1;4 5 6;1 2 3];
>> [V,D]=eig(A)
V =
   -0.3482   -0.8581    0.4082
   -0.8704    0.1907   -0.8165
   -0.3482    0.4767    0.4082
D =

    9.0000         0         0
         0    2.0000         0
         0         0   -0.0000
>> inv(V)*A*V
ans =
    9.0000   -0.0000   -0.0000
   -0.0000    2.0000   -0.0000
   -0.0000   -0.0000   -0.0000
>> V\A*V
ans =
    9.0000   -0.0000    0.0000
   -0.0000    2.0000         0
   -0.0000   -0.0000    0.0000
```

Modal Coordinates, 1

Consider the system in diagonal form. The z coordinates are referred to as 'modal coordinates'.

$$\dot{z}_i = \lambda_i z_i + \bar{b}_i u, \; i = 1, \ldots, n$$

Suppose we solve these equations to obtain $z_i(t), i = 1, \ldots, n,$ the we can obtain the solution in the original coordinates via

$$x(t) = Tz(t) = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = h_1 z_1(t) + h_2 z_2(t) + \cdots + h_n z_n(t)$$

Notice that each term $h_i z_i(t)$ describes a motion that takes place in the line defined by h_i.

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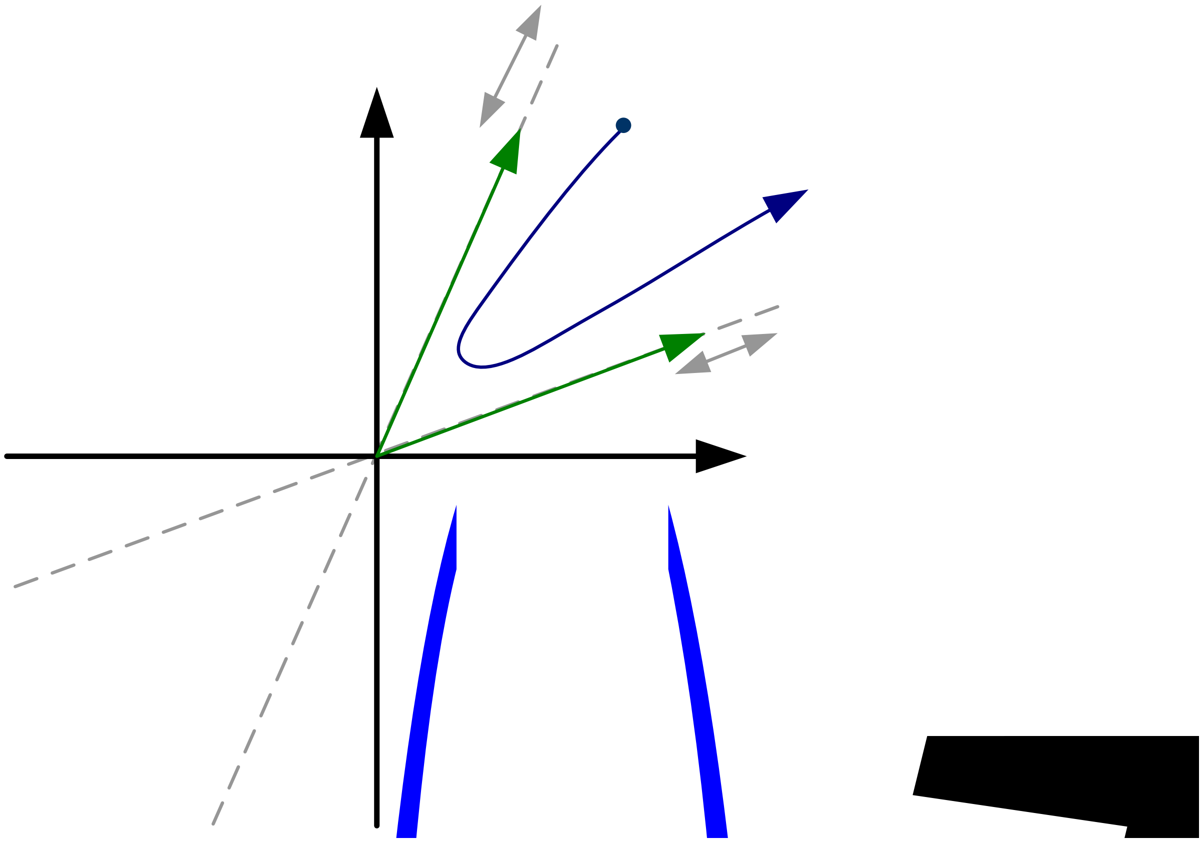
Model Coordinates, 2



Modal Coordinates, 3

Suppose the system is unforce, $u(t) = 0$

The initial conditions for the modal coordinates are $z_0 = T^{-1} x_0$

The solution is

$$z_i(t) = e^{\lambda_i t} z_{0,i},\ i = 1,\ldots,n$$

$$x(t) = h_1 e^{\lambda_1 t} z_{0,1} + \cdots + h_n e^{\lambda_n t} z_{0,n}$$

The modes are the vector time functions $h_i e^{\lambda_i t}, h_i$ is the mode shape.

The solution is a linear combination of the modes.

If λ_i is real, so is h_i and the modal response is a simple exponential.

If there are n linearly independent eigenvectors, then the set of solutions

$$\left\{ h_1 e^{\lambda_1 t}, \ldots, h_n e^{\lambda_n t} \right\}$$

is called a fundamental set of solutions. Any solution to the homogeneous equation $\dot{x} = Ax$ is a linear combination of the fundamental solutions.

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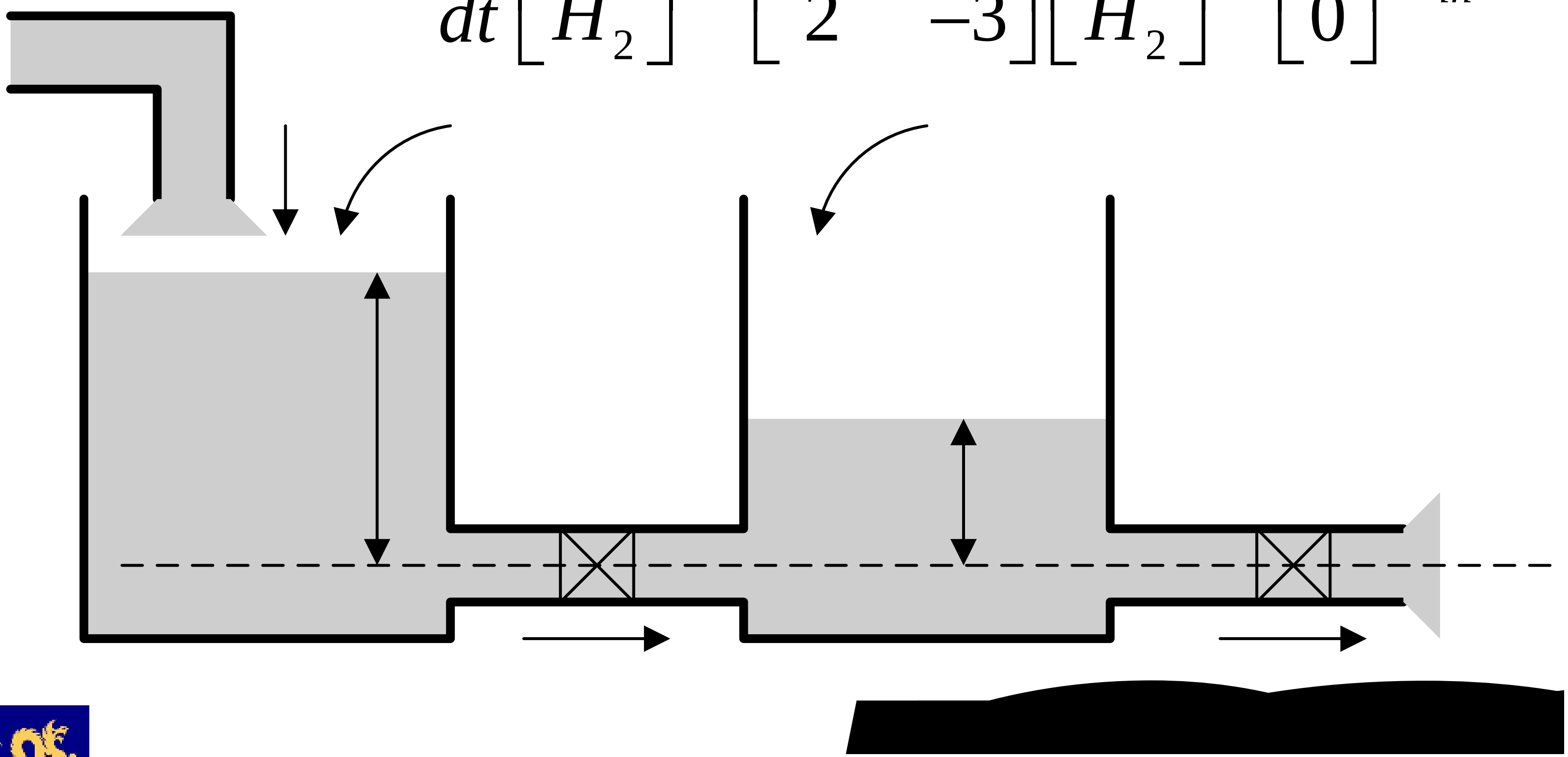
Example

$$A_1 = 1,\ A_2 = 1/2,\ R_1 = R_2 = 1$$

$$\frac{d}{dt}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} Q_{in}$$



Example Cont'd

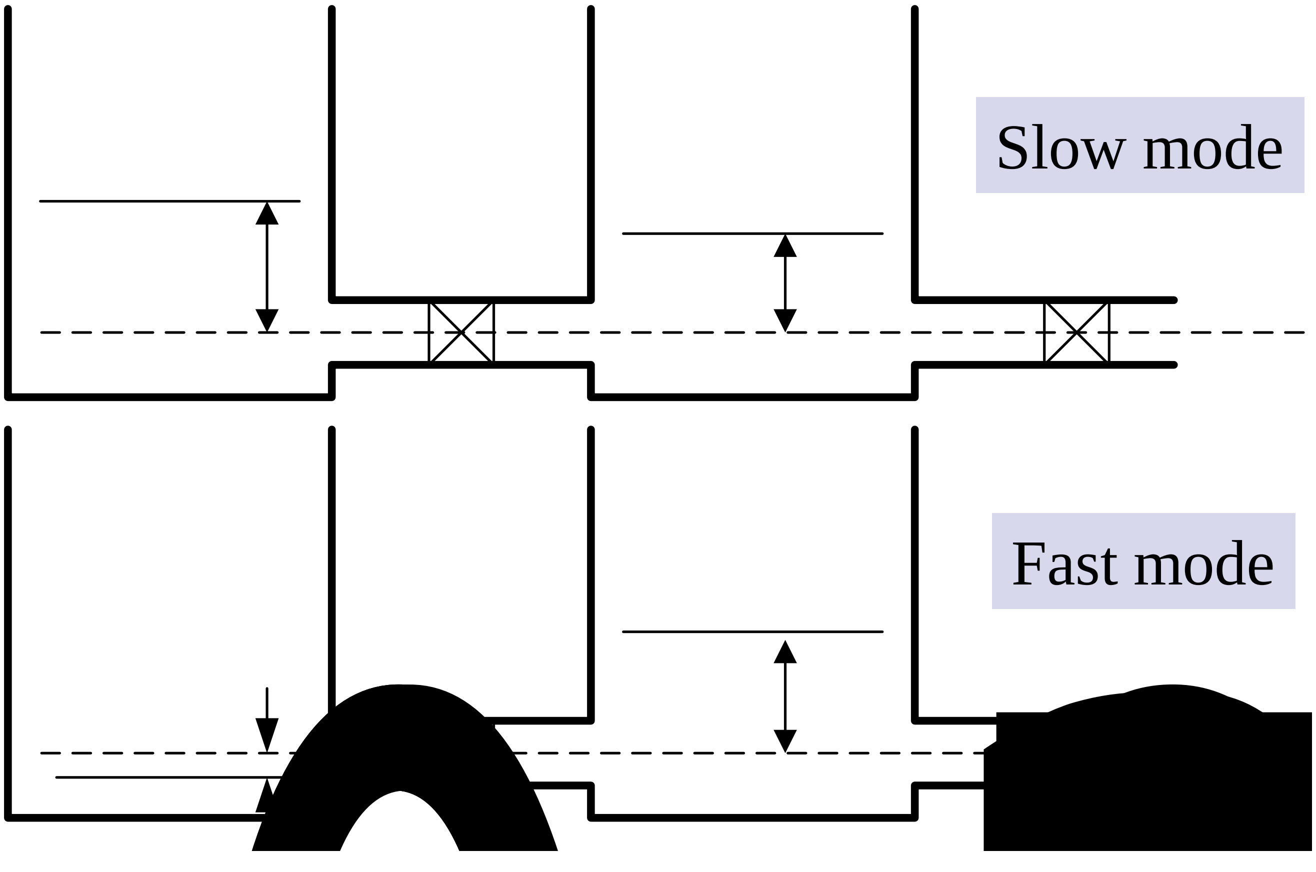
```
>> A=[-1 1;2 -3];
>> [E,V]=eig(A)
E =
    0.8069   -0.3437
    0.5907    0.9391
V =

   -0.2679         0
         0   -3.7321
```

$$\text{slow mode: } \lambda_1 = -0.2679, h_1 = \begin{bmatrix} 0.8069 \\ 0.5907 \end{bmatrix}$$

$$\text{fast mode: } \lambda_2 = -3.7321, h_2 = \begin{bmatrix} -0.3437 \\ 0.9391 \end{bmatrix}$$

Example, Cont'd



Complex Modes, 1

If the an eigenvalue λ_i is complex, the so is the eigenvector h_i. To make things easier to interpret, we can construct real ones. Suppose $\lambda_1 = \sigma + j\omega$ is a complex eigenvalue, and $\lambda_2 = \lambda_1^*$. Then the corresponding eigenvectors are $h_1, h_2 = h_1^*$. Correspondly, the two fundamental solutions are $h_1 e^{\lambda_1 t}, h_1^* e^{\lambda_1^* t}$. They are complex. We will replace them with real ones.

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Complex Modes, 2

Define

$$x_1(t) = \frac{1}{2}\left(h_1 e^{\lambda_1 t} + h_1^* e^{\lambda_1^* t}\right)$$

$$= \frac{1}{2}\begin{pmatrix} \left(h_{1,R} + jh_{1,I}\right)e^{\sigma t}\left(\cos\omega t + j\sin\omega t\right) \\ + \left(h_{1,R} - jh_{1,I}\right)e^{\sigma t}\left(\cos\omega t - j\sin\omega t\right) \end{pmatrix}$$

$$= h_{1,R}e^{\sigma t}\cos\omega t - h_{1,I}e^{\sigma t}\sin\omega t$$

$$x_2(t) = \frac{1}{2}\left(-h_1 e^{\lambda_1 t} + h_1^* e^{\lambda_1^* t}\right) = h_{1,R}e^{\sigma t}\sin\omega t + h_{1,I}e^{\sigma t}\cos\omega t$$

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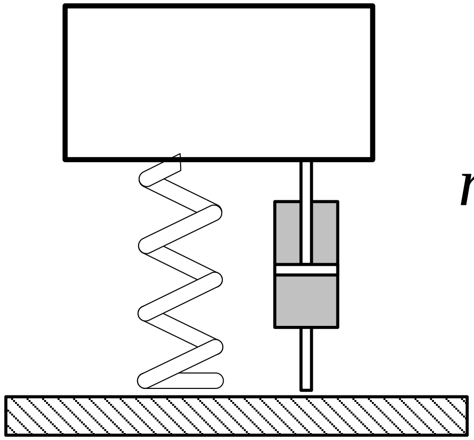
Example



$$\dot{x} = v$$

$$m\dot{v} = -cv - kx$$

$$m = 1, k = 4, c = 1/2$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -0.5 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$h_{1,R} = \begin{bmatrix} 0.0559 \\ -0.8944 \end{bmatrix},$$

$$h_{1,I} = \begin{bmatrix} 0.4437 \\ 0 \end{bmatrix}$$

```
>> A=[0 1;-4 -0.5];
>> [E,V]=eig(A)
E =
   0.0559 + 0.4437i   0.0559 - 0.4437i
  -0.8944            -0.8944
V =
  -0.2500 + 1.9843i        0
       0             -0.2500 - 1.9843i
```

Lateral Dynamics: Boeing 747 (cruise @ 40,000 ft)

```
>> A=[-0.0558   -0.997    0.0802   0.0415
       0.598    -0.1150  -0.0318   0
      -3.05      0.388   -0.465    0
       0.        0.0805   1.        0];
```

$$\begin{bmatrix} \beta \\ r \\ p \\ \phi \end{bmatrix}$$

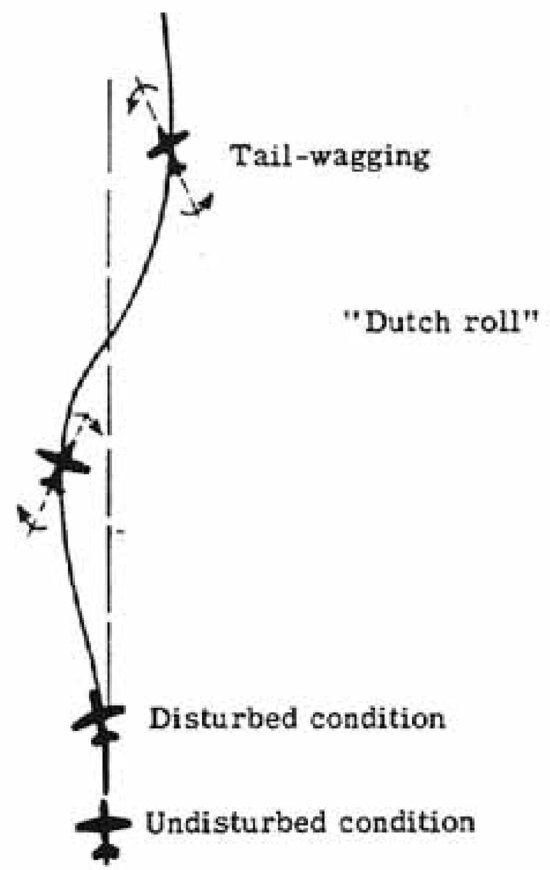
Eigenvectors

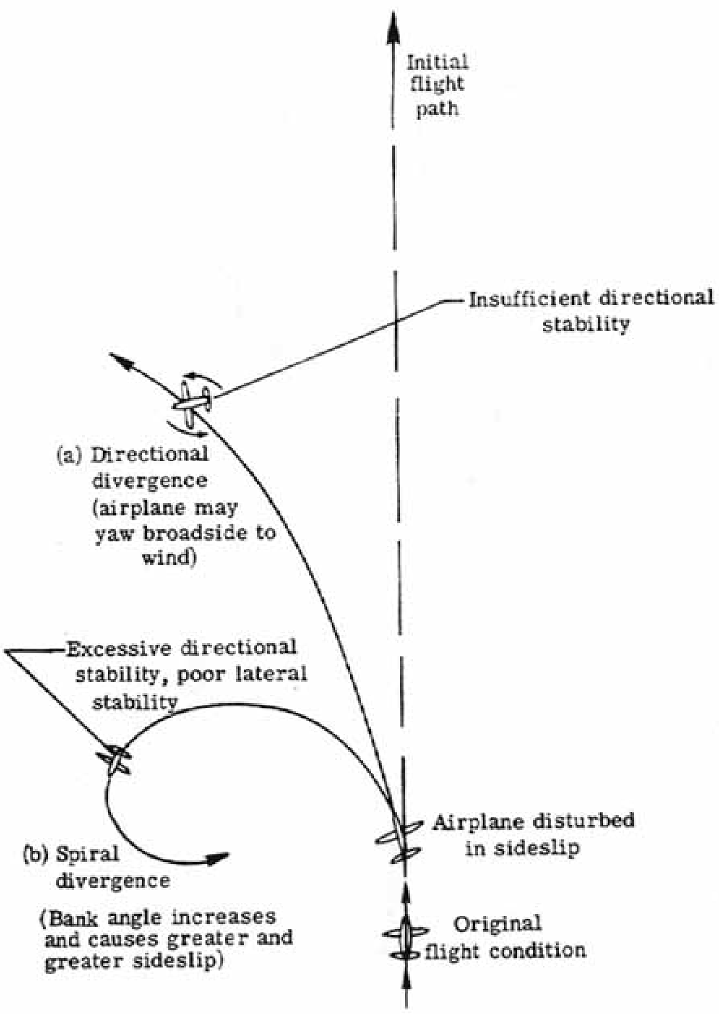
0.1994 - 0.1063i	0.1994 + 0.1063i	-0.0172	0.0067
-0.0780 - 0.1333i	-0.0780 + 0.1333i	-0.0118	0.0404
-0.0165 + 0.6668i	-0.0165 - 0.6668i	-0.4895	-0.0105
0.6930	0.6930	0.8717	0.9991

Eigenvalues

-0.0329 + 0.9467i	-0.0329 - 0.9467i	-0.5627	-0.0073
Dutch roll		Roll	Spiral

Dutch Roll & Spiral Modes





Longitudinal Dynamics: Boeing 747 (cruise@20,000ft)

$$\begin{bmatrix} u \\ v \\ q \\ \theta \end{bmatrix}$$

```
>> A=[-0.00643    0.0263     0      -32.2
      -0.0941     -0.624     820      0
      -0.000222   -0.00153   -0.668   0
       0           0          1       0];
```

Eigenvectors

-0.0124 + 0.0040i	-0.0124 - 0.0040i	0.9894	0.9894
-0.9999	-0.9999	-0.1451 + 0.0005i	-0.1451 - 0.0005i
0.0000 - 0.0014i	0.0000 + 0.0014i	0.0000 - 0.0000i	0.0000 + 0.0000i
-0.0009 + 0.0005i	-0.0009 - 0.0005i	-0.0002 - 0.0003i	-0.0002 + 0.0003i

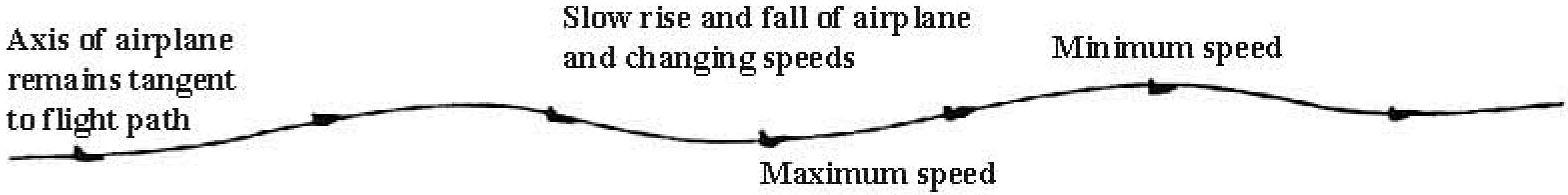
Eigenvalues

-0.6463 + 1.1211i	-0.6463 - 1.1211i	-0.0030 + 0.0098i	-0.0030 - 0.0098i

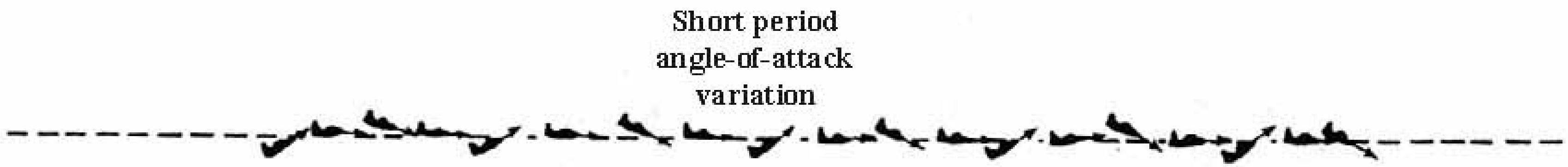
Short period

Phugoid

Phugoid and Short-period Modes



(a) Phugoid longitudinal oscillation.



(b) Short-period longitudinal oscillation.

Longitudinal Dynamics AFTI-16 (landing Configuration)

```
>> A=[-0.0507      -3.861    0      -32.2
      -0.00117     -0.5164   1       0
      -0.000129     1.4168  -0.4932  0
       0            0        1       0];
```

$$\begin{bmatrix} \Delta V \\ \Delta \alpha \\ q \\ \theta \end{bmatrix}$$

0.9943	0.9995	1.0000	1.0000
0.0633	-0.0142	0.0005 + 0.0003	0.0005 - 0.0003i
-0.0740	-0.0165	0.0013 + 0.0002i	0.0013 - 0.0002i
0.0434	-0.0226	-0.0003 - 0.0064i	-0.0003 + 0.0064i

-1.7036	0.7310	-0.0438 + 0.2066i	-0.0438 - 0.2066i

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Summary

- Similarity transformations
- Diagonalization using Eigenvectors
- Modes & Modal coordinates
- Interpreting behavior in terms of modes
- Converting complex modes into real ones
- Lateral – Dutch roll and spiral modes
- Longitudinal – phugoid and short period modes